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A THEOREM CONCERNING UNIFORM
SIMPLIFICATION AT A TRANSITION POINT
AND THE PROBLEM OF RESONANCE

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A THEOREM CONCERNING UNIFORM SIMPLIFICATION AT A TRANSITION

POINT AND THE PROBLEM OF RESONANCE*

Yasutaka Sibuya[†]

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ABSTRACT

Given sectors $S_j = \{\varepsilon; a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho\}$ ($1 \leq j \leq v$) and functions δ_j ($1 \leq j \leq v$) such that (i) $\bigcup_j S_j = \{\varepsilon; 0 < |\varepsilon| < \rho\}$, (ii) δ_j is holomorphic in S_j , (iii) δ_j is asymptotically zero as $\varepsilon \rightarrow 0$ in S_j , (iv) $|\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp\{-c_1/|\varepsilon|^\lambda\}$ in $S_j \cap S_k$ for some positive numbers c_0, c_1 and λ whenever $S_j \cap S_k \neq \emptyset$, we prove that $|\delta_j(\varepsilon)| \leq c_2 \exp\{-c_1/|\varepsilon|^\lambda\}$ in S_j for some positive number c_2 . Then, utilizing this result, we prove that Matkowsky-condition implies the resonance in the sense of N. Kopell under a reasonable assumption. The sufficiency of Matkowsky-condition with regard to the Ackerberg-O'Malley resonance has been an open question. This work gives an affirmative answer to this question in a reasonably general case.

AMS (MOS) Subject Classifications: 30B40, 30E15, 33A40, 34E20

Key Words: Analytic continuation, Asymptotic representations in the complex domain, Parabolic cylinder functions, Singular perturbations, Turning point theory

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

The basic question is as follows:

Consider a second-order linear differential equation:

$$(1) \quad \epsilon \frac{d^2 v}{dx^2} + F(x, \epsilon) \frac{dv}{dx} + G(x, \epsilon) v = 0$$

under some reasonable assumptions on F and G . Let $v(x, \epsilon)$ be a solution.

Then, generally speaking, $\lim_{\epsilon \rightarrow 0} v(x, \epsilon)$ satisfies the first order equation:

$$(2) \quad F(x, 0) \frac{dv}{dx} + G(x, 0) v = 0.$$

The problem of finding the relation between (1) and (2) is called the problem of singular perturbations. "Singular" means that solutions of the equation (2) do not contain as many free parameters as solutions of (1) do. In other words, $\lim_{\epsilon \rightarrow 0} v(x, \epsilon)$ loses something. This phenomenon is explained by means of "boundary-layer" theory. Actually, various physical phenomena exhibit similar behaviours. Therefore, the problem of singular perturbations has been studied for many years.

→ In a certain situation which arises naturally in applications, $\lim_{\epsilon \rightarrow 0} v(x, \epsilon) = 0$ identically for practically all the solutions v of (1), except when F and G are related in a specific way. This exceptional case is called the case of resonance. It is important to find an effectively computable condition for the resonance. B. J. Matkowsky found such a condition. However, so far, it has been mathematically very difficult to prove that the Matkowsky-condition actually guarantees the resonance. The difficulty is due to the fact that a quantity which is decisive in determining the resonance is so small that any existing mathematical tool has failed to dig this quantity out of the differential equation clearly. In this work, we shall provide such a tool.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A THEOREM CONCERNING UNIFORM SIMPLIFICATION AT A TRANSITION
POINT AND THE PROBLEM OF RESONANCE*

Yasutaka Sibuya[†]

1. Introduction: The main result of this paper is the following theorem:

Theorem 1.1: Let

$$(1.1) \quad S_j = \{\varepsilon; a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho\} \quad (j = 1, \dots, \nu)$$

be sectors in the complex ε -plane, where ρ is a positive number and the a 's and the b 's are real numbers. Let $\delta_1(\varepsilon), \dots, \delta_\nu(\varepsilon)$ be functions of ε .

Assume that

- (i) $S_1 \cup S_2 \cup \dots \cup S_\nu = \{\varepsilon; 0 < |\varepsilon| < \rho\}$;
- (ii) $\delta_j(\varepsilon)$ is holomorphic in S_j ;
- (iii) $\delta_j(\varepsilon)$ is asymptotically zero as $\varepsilon \rightarrow 0$ in S_j , i.e.
 $|\delta_j(\varepsilon)| \leq K_N |\varepsilon|^N \quad (N = 0, 1, \dots) \text{ in } S_j$

for some positive numbers K_N ;

- (iv) if $S_j \cap S_k \neq \emptyset$, we have

$$(1.2) \quad |\delta_j(\varepsilon) - \delta_k(\varepsilon)| \leq c_0 \exp(-c_1/|\varepsilon|^\lambda) \text{ in } S_j \cap S_k,$$

for some positive numbers c_0, c_1 and λ .

Then, there exists a positive number H such that

$$(1.3) \quad |\delta_j(\varepsilon)| \leq H \exp(-c_1/|\varepsilon|^\lambda) \text{ in } S_j, \quad j = 1, 2, \dots, \nu.$$

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We shall prove this theorem in Section 8. (For another proof, see J.-P. Ramis [5; Theorem 11-(i), p. 189].) In other sections, utilizing Theorem 1.1, we shall treat the following problem:

We consider a differential equation:

$$(1.4) \quad \varepsilon d^2 v / dx^2 + F(x, \varepsilon) dv / dx + G(x, \varepsilon) v = 0 ,$$

where F and G are holomorphic in two complex variables x and ε in a domain:

$$(1.5) \quad x \in D_0, \quad |\varepsilon| < \rho_0 ,$$

where D_0 is a domain in the x -plane and ρ_0 is a positive number. We assume that D_0 contains a real interval:

$$(1.6) \quad I_0 = \{x; -a \leq \operatorname{Re}(x) \leq b, \operatorname{Im}(x) = 0\} ,$$

where a and b are positive numbers. We also assume that

$$(1.7) \quad F(x, 0) = -2x .$$

We say that the differential equation (1.4) satisfies Matkowsky-condition, if there exists a non-trivial formal power series solution of (1.4):

$$(1.8) \quad v = \sum_{m=0}^{\infty} a_m(x) \varepsilon^m$$

such that all the $a_m(x)$ are bounded on the real interval I_0 . We also say that the differential equation (1.4) exhibits a resonance in the sense of N. Kopell on I_0 if there exists a solution $v(x, \varepsilon)$ satisfying $v(b, \varepsilon) = 1$, such that $v(x, \varepsilon)$ converges uniformly on I_0 as $\varepsilon \rightarrow +0$ to a non-trivial solution of

$$(1.9) \quad F(x, 0) dv / dx + G(x, 0) v = 0 .$$

(Cf. B. J. Matkowsky [4] and N. Kopell [2].)

We shall prove the following theorem:

THEOREM 1.2. If \mathcal{D}_0 is a disk with the center at $x = 0$, i.e.

$$(1.10) \quad \mathcal{D}_0 = \{x; |x| < r_0\} \text{ for some } r_0 > 0.$$

Then, Matkowsky-condition implies the resonance in the sense of N. Kopell.

In our argument, the assumption that F and G are holomorphic in (x, ε) in a poly-disk (1.5) is indispensable. In our proof, we follow roughly the guide-line given by R. McKelvey and R. Bohac [3]. It seems to us that our results yield a sharp estimate for eigenvalues studied by P. P. N. de Groen [1]. In Section 2, we discuss a more general case.

Throughout this research, the author enjoyed lively discussions with N. Kopell, B. J. Matkowsky and P. P. N. de Groen.

2. A standard form: Let ρ_0 be a positive number and let \mathcal{D} be a domain in the complex ξ -plane which contains a real interval

$$(2.1) \quad I = \{\xi; -\alpha \leq \operatorname{Re}(\xi) \leq \beta, \operatorname{Im}(\xi) = 0\},$$

where α and β are positive numbers.

We shall consider a linear differential equation:

$$(2.2) \quad \varepsilon d^2 v / d\xi^2 + f(\xi, \varepsilon) dv / d\xi + g(\xi, \varepsilon) v = 0,$$

where f and g are holomorphic in two variables ξ and ε in the domain

$$(2.3) \quad \xi \in \mathcal{D}, \quad |\varepsilon| < \rho_0.$$

Set

$$(2.4) \quad f_0(\xi) = f(\xi, 0).$$

We assume that

$$(2.5) \quad f_0(0) = 0, \quad f'_0(0) \neq 0,$$

$$(2.6) \quad \xi f_0(\xi) < 0 \text{ for } \xi \in I, \text{ if } \xi \neq 0.$$

Under this situation, we can write f_0 as

$$(2.7) \quad f_0(\xi) = \xi h(\xi),$$

where $h(\xi)$ is holomorphic in \mathcal{D} and

$$(2.8) \quad h(\xi) < 0 \text{ for } \xi \in I.$$

Let us change the independent variable by

$$(2.9) \quad x = \varphi(\xi) = \left\{ - \int_0^\xi f_0(t) dt \right\}^{1/2}.$$

Then, (2.2) becomes

$$(2.10) \quad \varepsilon d^2 v / dx^2 + F(x, \varepsilon) dv / dx + G(x, \varepsilon) v = 0,$$

where

$$(2.11) \quad F(\varphi, \varepsilon) = (\varphi')^{-2} \{ \varphi' f + \varepsilon \varphi'' \}, \quad G(\varphi, \varepsilon) = (\varphi')^{-2} g.$$

Since $f_0 = -2\varepsilon\varphi'$, we have

$$(2.12) \quad F(x, \varepsilon) = -2x + \varepsilon k(x, \varepsilon),$$

and $k(x, \epsilon)$ and $G(x, \epsilon)$ are holomorphic in a domain

$$(2.13) \quad x \in \mathcal{D}_0, \quad |\epsilon| < \rho_0,$$

where \mathcal{D}_0 is a domain in the x -plane which contains the real interval:

$$(2.14) \quad I_0 = \{x; -a \leq \operatorname{Re}(x) \leq b, \operatorname{Im}(x) = 0\},$$

where

$$(2.15) \quad a = \sqrt{-\int_0^{-a} f_0(t) dt}, \quad b = \sqrt{-\int_0^b f_0(t) dt}.$$

Another transformation:

$$(2.16) \quad v = w \exp\left\{-\frac{1}{2\epsilon} \int_0^x F(t, \epsilon) dt\right\}$$

takes (2.10) to

$$(2.17) \quad \epsilon^2 d^2 w / dx^2 - \left\{ \frac{1}{4} F(x, \epsilon)^2 + \epsilon \left(\frac{1}{2} \frac{\partial F}{\partial x}(x, \epsilon) - G(x, \epsilon) \right) \right\} w = 0.$$

Note that

$$(2.18) \quad \frac{1}{4} F^2 + \epsilon \left(\frac{1}{2} \frac{\partial F}{\partial x} - G \right) = x^2 + \epsilon R(x, \epsilon),$$

where R is holomorphic in (2.13).

Remark: To find the domain \mathcal{D}_0 , we must take into account not only singularities of f and g , but also singularities of φ , i.e. the transformation (2.9).

In particular, any zeros of f_0 would yield branch-points with respect to x .

3. Formal simplification: It is known that there exist three formal power series in ε :

$$(3.1) \quad A(x, \varepsilon) = \sum_{m=0}^{\infty} A_m(x) \varepsilon^m,$$

$$(3.2) \quad B(x, \varepsilon) = \sum_{m=0}^{\infty} B_m(x) \varepsilon^m,$$

and

$$(3.3) \quad C(\varepsilon) = \sum_{m=0}^{\infty} C_m \varepsilon^m$$

such that

- (i) $A_m(x)$ and $B_m(x)$ are holomorphic in the domain \mathcal{D}_0 ;
- (ii) C_m are constants;
- (iii) the formal transformation:

$$(3.4) \quad w = A(x, \varepsilon)u + B(x, \varepsilon)(\varepsilon du/dx)$$

takes (2.17) to

$$(3.5) \quad \varepsilon^2 d^2 u/dx^2 - \{x^2 + \varepsilon C(\varepsilon)\}u = 0;$$

- (iv) we have

$$(3.6) \quad A_0(x)^2 - (xB_0(x))^2 = 1 \text{ identically in } \mathcal{D}_0.$$

To effect the transformation (3.4), we differentiate both sides of (3.4) with respect to x . Then, we derive

$$(3.7) \quad \varepsilon dw/dx = (\varepsilon A' + (x^2 + \varepsilon C)B)u + (A + \varepsilon B')(\varepsilon du/dx),$$

and

$$(3.8) \quad \varepsilon^2 d^2 w/dx^2 = (\varepsilon(\varepsilon A' + (x^2 + \varepsilon C)B)' + (x^2 + \varepsilon C)(A + \varepsilon B'))u \\ + ((\varepsilon A' + (x^2 + \varepsilon C)B) + \varepsilon(A + \varepsilon B'))(\varepsilon du/dx),$$

where $'$ denotes $\partial/\partial x$. Since $\varepsilon^2 d^2 w/dx^2 = (x^2 + \varepsilon R)w$, we derive the following equations on A , B and C :

$$(3.9) \quad \begin{cases} (x^2 + \varepsilon R)A = \varepsilon(\varepsilon A' + (x^2 + \varepsilon C)B)' + (x^2 + \varepsilon C)(A + \varepsilon B') , \\ (x^2 + \varepsilon R)B = (\varepsilon A' + (x^2 + \varepsilon C)B) + \varepsilon(A + \varepsilon B')' . \end{cases}$$

In particular, if we put

$$X = A_0, \quad Y = xB_0 ,$$

we have

$$dX/dx = \frac{R_0(x) - C_0}{2x} Y, \quad dY/dx = \frac{R_0(x) - C_0}{2x} X ,$$

where $R_0(x) = R(x, 0)$. Hence

$$d(x^2 - Y^2)/dx = 0 \text{ identically.}$$

Choose $C_0 = R_0(0)$ and the initial condition: $X(0) = 1, Y(0) = 0$. Then, we can determine A_0, B_0 and C_0 so that (3.6) is satisfied. Other coefficients A_m, B_m and C_m can be determined in a similar way.

By virtue of (3.6), we can solve (3.4) and (3.7) with respect to u and $\varepsilon du/dx$:

$$(3.10) \quad \begin{cases} u = E_{11}(x, \varepsilon)w + E_{12}(x, \varepsilon)(\varepsilon dw/dx) , \\ \varepsilon du/dx = E_{21}(x, \varepsilon)w + E_{22}(x, \varepsilon)(\varepsilon dw/dx) , \end{cases}$$

where E_{jk} are formal power series in ε whose coefficients are holomorphic in \mathcal{D}_0 . In particular,

$$(3.11) \quad \begin{cases} E_{11}(x, 0) = E_{22}(x, 0) = A_0(x) , \\ E_{12}(x, 0) = -B_0(x), \quad E_{21}(x, 0) = -x^2 B_0(x) . \end{cases}$$

Note that

$$(3.12) \quad C_0 = R_0(0) = -1 + 2 \frac{g(0, 0)}{f'_0(0)} .$$

4. Outer expansions: A formal power series in ϵ :

$$(4.1) \quad v = \sum_{m=0}^{\infty} a_m(x) \epsilon^m$$

is called an outer expansion associated with the differential equation (2.10), if (4.1) formally satisfies (2.10). The power series (4.1) is an outer expansion if and only if

$$(4.2) \quad \begin{cases} -2x \, da_0/dx + G_0(x) a_0 = 0, \\ -2x \, da_m/dx + G_0(x) a_m = L_m(x) - d^2 a_{m-1}(x)/dx^2 \quad (m \geq 1), \end{cases}$$

where $G_0(x) = G(x, 0)$ and $L_m(x)$ is linear homogeneous in a_0, \dots, a_{m-1} and $da_0/dx, \dots, da_{m-1}/dx$ with coefficients holomorphic in \mathcal{D}_0 .

DEFINITION 4.1: The differential equation (2.10) is said to satisfy Matkowsky-condition, if there exists a non-trivial outer expansion (4.1) such that all the $a_m(x)$ are bounded on the real interval I_0 (cf. (2.14)).

LEMMA 4.2: The differential equation (2.10) satisfies Matkowsky-condition if and only if C_0 is a negative odd integer and

$$(4.3) \quad C_m = 0 \quad (m \geq 1).$$

Proof: The transformation

$$(4.4) \quad u = y \exp\{-x^2/(2\epsilon)\}$$

changes (3.5) to

$$(4.5) \quad \epsilon d^2 y/dx^2 - 2x \, dy/dx - (1 + C)y = 0.$$

By a straight-forward computation, we can prove that the differential equation (4.5) satisfies Matkowsky-condition if and only if C_0 is a negative odd integer and $C_m = 0$ for $m \geq 1$.

Note also that, if all the a_m are bounded, then all the da_m/dx are bounded. Otherwise, $d^2 a_m/dx^2$ would have much worse singularities at $x = 0$, and hence a_{m+1} would be unbounded (cf. (4.2)).

Finally, by manipulating with the transformations (2.16), (3.4) and (3.7), and (3.10) together with (4.4), we can show that the differential equation (2.10) satisfies Matkowsky-condition if and only if the differential equation (4.5) satisfies the same condition. This completes the proof of Lemma 4.2.

5. Uniform simplification: Hereafter, we shall assume that

$$(5.1) \quad C_0 = -p, \text{ where } p \text{ is a positive odd integer;}$$

$$(5.2) \quad C_m = 0 \text{ for } m \geq 1;$$

$$(5.3) \quad D_0 = \{x; |x| < r_0\} \text{ for some } r_0 > 0.$$

The assumption (5.3) means that D_0 is a disk of radius r_0 with the center at $x = 0$.

Let us choose two positive numbers r_1 and r such that

$$(5.4) \quad 0 < r_1 < r < r_0$$

and that the disk

$$(5.5) \quad D_1 = \{x; |x| < r_1\}$$

contains the real interval I_0 (cf. (2.14)).

Let us denote by $T(x, \epsilon)$ the two-by-two matrix:

$$(5.6) \quad \begin{bmatrix} A(x, \epsilon) & B(x, \epsilon) \\ \epsilon A'(x, \epsilon) + (x^2 - \epsilon p)B(x, \epsilon) & A(x, \epsilon) + \epsilon B'(x, \epsilon) \end{bmatrix}$$

(cf. (3.4) and (3.7)). Set

$$(5.7) \quad U = \begin{bmatrix} u \\ \epsilon du/dx \end{bmatrix}, \quad W = \begin{bmatrix} w \\ \epsilon dw/dx \end{bmatrix}.$$

Then, the formal transformation

$$(5.8) \quad W = T(x, \epsilon)U$$

takes the system

$$(5.9) \quad \epsilon dW/dx = \begin{bmatrix} 0 & 1 \\ x^2 + \epsilon R(x, \epsilon) & 0 \end{bmatrix} W$$

to

$$(5.10) \quad \epsilon dU/dx = \begin{bmatrix} 0 & 1 \\ x^2 - \epsilon p & 0 \end{bmatrix} U.$$

The inverse of the matrix $T(x, \epsilon)$ is given by

$$(5.11) \quad T(x, \epsilon)^{-1} = \begin{bmatrix} E_{11}(x, \epsilon) & E_{12}(x, \epsilon) \\ E_{21}(x, \epsilon) & E_{22}(x, \epsilon) \end{bmatrix}$$

(cf. (3.10)).

Set

$$(5.12) \quad \mathcal{D}_2 = \{x; |x| < r\}.$$

It is known that there exist two positive numbers ρ_1 and ρ_2 , a function $\delta(\epsilon)$, and a two-by-two matrix $P(x, \epsilon)$ such that

(i) $\delta(\epsilon)$ is holomorphic in the sector

$$(5.13) \quad S = \{\epsilon; |\arg \epsilon| < \rho_1, 0 < |\epsilon| < \rho_2\};$$

(ii) $\delta(\epsilon)$ is asymptotically zero as $\epsilon \rightarrow 0$ in S , i.e.

$$(5.14) \quad |\delta(\epsilon)| \leq K_N |\epsilon|^N \quad (N = 0, 1, 2, \dots) \text{ in } S$$

for some positive numbers K_N ;

(iii) entries of P and P^{-1} are holomorphic in the domain

$$(5.15) \quad x \in \mathcal{D}_2, \quad \epsilon \in S;$$

(iv) P (resp. P^{-1}) admits the matrix T (resp. T^{-1}) as an asymptotic expansion as $\epsilon \rightarrow 0$ in S which is valid uniformly in \mathcal{D}_2 ;

(v) the transformation

$$(5.16) \quad W = P(x, \epsilon)V$$

takes (5.9) to

$$(5.17) \quad \epsilon dV/dx = \begin{bmatrix} 0 & 1 \\ x^2 - \epsilon(p + \delta(\epsilon)) & 0 \end{bmatrix} V$$

in the domain (5.15). (Cf. Y. Sibuya [6].)

Utilizing this result and manipulating with rotations of the disk \mathcal{D}_2 , we can prove the following lemma:

LEMMA 5.1: There exist sectors

$$(5.18-j) \quad S_j = \{\epsilon; a_j < \arg \epsilon < b_j, \quad 0 < |\epsilon| < \rho_3\} \quad (j = 1, 2, \dots, k),$$

(where ρ_3 is a positive number and the a's and the b's are real numbers), functions $\delta_1(\epsilon), \dots, \delta_k(\epsilon)$, and two-by-two matrices $P_1(x, \epsilon), \dots, P_k(x, \epsilon)$ such that $S_1 \cup \dots \cup S_k = \{\epsilon; 0 < |\epsilon| < \rho_3\}$ and that

(i) $\delta_j(\epsilon)$ is holomorphic in S_j ;

(ii) $\delta_j(\epsilon)$ is asymptotically zero as $\epsilon \rightarrow 0$ in S_j ;

(iii) entries of P_j and P_j^{-1} are holomorphic in the domain

$$(5.19-j) \quad x \in D_2, \quad \epsilon \in S_j;$$

(iv) P_j (resp. P_j^{-1}) admits the matrix T (resp. T^{-1}) as an asymptotic expansion as $\epsilon \rightarrow 0$ in S_j which is valid uniformly in D_2 ;

(v) the transformation

$$(5.20) \quad W = P_j(x, \epsilon)V_j$$

takes (5.9) to

$$(5.21-j) \quad \epsilon dV_j/dx = \begin{bmatrix} 0 & 1 \\ x^2 - \epsilon(p + \delta_j(\epsilon)) & 0 \end{bmatrix} V_j$$

in the domain (5.19-j).

6. An estimate for $\delta_j(\epsilon)$. In this section, as an application of our main theorem (cf. Theorem 1.1), we shall derive an estimate

$$(6.1) \quad |\delta_j(\epsilon)| \leq H_j \exp(-r^2/|\epsilon|) \quad \text{for } \epsilon \in S_j,$$

where H_j is a positive number. To do this, it is sufficient to prove that, if $S_k \cap S_j \neq \emptyset$, we have

$$(6.2) \quad |\delta_k(\epsilon) - \delta_j(\epsilon)| \leq M_{kj} \exp(-r^2/|\epsilon|) \quad \text{for } \epsilon \in S_k \cap S_j,$$

where M_{kj} is a positive number. To derive an estimate (6.2), we need some preparation.

Let us consider the differential equation

$$(6.3) \quad d^2z/dt^2 - (t^2 - a)z = 0, \quad \text{where } a \text{ is a parameter.}$$

This equation admits a solution

$$(6.4) \quad z = Z(t, a)$$

such that

(i) Z is an entire function in (t, a) ;

$$(ii) \quad \lim_{t \rightarrow +\infty} t^{\frac{1}{2}(1-a)} e^{\frac{1}{2}t^2} Z(t, a) = 1$$

uniformly in a if a is in a compact set in the a -plane.

The solution $Z(t, a)$ is uniquely determined by (i) and (ii). The functions $Z((-i)t, -a)$, $Z(-t, a)$, and $Z(it, -a)$ are also solutions of (6.3). Set

$$(6.5-0) \quad \psi_0(t, a) = \begin{bmatrix} Z(t, a) & Z((-i)t, -a) \\ Z'(t, a) & (-i)Z'((-i)t, -a) \end{bmatrix},$$

$$(6.5-1) \quad \psi_1(t, a) = \begin{bmatrix} Z((-i)t, -a) & Z(-t, a) \\ (-i)Z'((-i)t, -a) & -Z'(-t, a) \end{bmatrix},$$

$$(6.5-2) \quad \psi_2(t, a) = \begin{bmatrix} Z(-t, a) & Z(it, -a) \\ -Z'(-t, a) & iZ'(it, -a) \end{bmatrix},$$

and

$$(6.5-(-1)) \quad \Psi_{-1}(t, a) = \begin{bmatrix} Z(it, -a) & Z(t, a) \\ iZ'(it, -a) & Z'(t, a) \end{bmatrix},$$

where ' denotes d/dt . These four matrices are matrices of independent solutions of (6.3).

Set

$$(6.6) \quad \lambda_1(a) = 2^{-\frac{1}{2}a} e^{\frac{1}{2}i\pi(a+1)} \frac{\sqrt{2\pi}}{\Gamma(1-a)}, \quad \lambda_2(a) = (-i)e^{\frac{1}{2}a\pi i},$$

and

$$(6.7) \quad C(a) = \begin{bmatrix} \lambda_1(a) & 1 \\ \lambda_2(a) & 0 \end{bmatrix}.$$

Then,

$$(6.8) \quad \begin{cases} \Psi_0(t, a) = \Psi_1(t, a)C(a), & \Psi_1(t, a) = \Psi_2(t, a)C(-a), \\ \Psi_2(t, a) = \Psi_{-1}(t, a)C(a), & \Psi_{-1}(t, a) = \Psi_0(t, a)C(-a). \end{cases}$$

Fix i and j so that $S_i \cap S_j \neq \emptyset$. Choose a branch of $\sqrt{\cdot}$ in the sector $S_i \cap S_j$. Set

$$(6.9) \quad \Delta(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^{\frac{1}{2}} \end{bmatrix},$$

and

$$(6.10-h) \quad \begin{cases} \hat{\Psi}_{i,h}(x, \varepsilon) = \Delta(\varepsilon) \Psi_h(x/\varepsilon^{\frac{1}{2}}, p + \delta_i(\varepsilon)), \\ \hat{\Psi}_{j,h}(x, \varepsilon) = \Delta(\varepsilon) \Psi_h(x/\varepsilon^{\frac{1}{2}}, p + \delta_j(\varepsilon)), \end{cases} \quad (h = -1, 0, 1, 2).$$

Then, $\hat{\Psi}_{i,0}(x, \varepsilon)$, $\hat{\Psi}_{i,1}(x, \varepsilon)$, $\hat{\Psi}_{i,2}(x, \varepsilon)$ and $\hat{\Psi}_{i,-1}(x, \varepsilon)$ (resp. $\hat{\Psi}_{j,0}(x, \varepsilon)$,

$\hat{\Psi}_{j,1}(x, \varepsilon)$, $\hat{\Psi}_{j,2}(x, \varepsilon)$ and $\hat{\Psi}_{j,-1}(x, \varepsilon)$) are fundamental matrix solutions of

(5.21-1) (resp. (5.21-4)) such that

$$(6.11-l) \quad \begin{cases} \phi_{l,0}(x,\varepsilon) = \phi_{l,1}(x,\varepsilon)C(p + \delta_l(\varepsilon)) , \\ \phi_{l,1}(x,\varepsilon) = \phi_{l,2}(x,\varepsilon)C(-p - \delta_l(\varepsilon)) , \\ \phi_{l,2}(x,\varepsilon) = \phi_{l,-1}(x,\varepsilon)C(p + \delta_l(\varepsilon)) , \\ \phi_{l,-1}(x,\varepsilon) = \phi_{l,0}(x,\varepsilon)C(-p - \delta_l(\varepsilon)) , \end{cases}$$

and

$$(6.11-j) \quad \begin{cases} \phi_{j,0}(x,\varepsilon) = \phi_{j,1}(x,\varepsilon)C(p + \delta_j(\varepsilon)) , \\ \phi_{j,1}(x,\varepsilon) = \phi_{j,2}(x,\varepsilon)C(-p - \delta_j(\varepsilon)) , \\ \phi_{j,2}(x,\varepsilon) = \phi_{j,-1}(x,\varepsilon)C(p + \delta_j(\varepsilon)) , \\ \phi_{j,-1}(x,\varepsilon) = \phi_{j,0}(x,\varepsilon)C(-p - \delta_j(\varepsilon)) . \end{cases}$$

Set

$$(6.12) \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,$$

and

$$(6.13) \quad \begin{cases} Q_{l,h}(x,\varepsilon) = \phi_{l,h}(x,\varepsilon) \exp\left\{(-1)^h \frac{x^2}{2\varepsilon} J\right\} , \\ Q_{j,h}(x,\varepsilon) = \phi_{j,h}(x,\varepsilon) \exp\left\{(-1)^h \frac{x^2}{2\varepsilon} J\right\} , \end{cases} \quad (h = -1, 0, 1, 2) .$$

It is known that, if (x,ε) is in a domain

$$(6.14-h) \quad x \in D_2, \quad \varepsilon \in S_i \cap S_j, \quad \left| \arg\left(\frac{x}{\varepsilon}\right) - \frac{1}{4}\pi - \frac{1}{2}h\pi \right| \leq \frac{1}{2}\pi - \nu ,$$

where ν is a small positive number, we have

$$(6.15) \quad \|Q_{j,h}(x,\varepsilon)\| \leq H|\varepsilon|^q, \quad \|Q_{j,h}(x,\varepsilon)^{-1}\| \leq H|\varepsilon|^q ,$$

where H is a positive number depending on ν , q is a real number, and $\| \cdot \|$ denotes a usual norm of matrices. Furthermore, the matrix

$$(6.16) \quad Q_{j,h}(x,\varepsilon) - Q_{l,h}(x,\varepsilon)$$

is asymptotically zero as $\varepsilon \rightarrow 0$ in $S_i \cap S_j$ uniformly in the domain

$$(6.14-h). \quad (\text{For those results, see, for example, V. Sibuya [6,7].})$$

Let $P_k(x, \epsilon)$ and $P_j(x, \epsilon)$ be the matrices given in Lemma 5.1. Then, $P_k(x, \epsilon)\phi_{k,0}(x, \epsilon)$ and $P_j(x, \epsilon)\phi_{j,0}(x, \epsilon)$ are two fundamental matrix solutions of (5.9) in the domain

$$(6.17) \quad x \in D_2, \quad \epsilon \in S_k \cap S_j.$$

Therefore, there exists a two-by-two matrix $L(\epsilon)$ such that

$$(6.18) \quad P_k(x, \epsilon)\phi_{k,0}(x, \epsilon) = P_j(x, \epsilon)\phi_{j,0}(x, \epsilon)L(\epsilon).$$

Note that $L(\epsilon)$ does not depend on x . It follows from (6.18) that

$$(6.19) \quad \exp\left\{-\frac{x^2}{2\epsilon}J\right\}L(\epsilon)\exp\left\{\frac{x^2}{2\epsilon}J\right\} = Q_{j,0}(x, \epsilon)^{-1}P_j(x, \epsilon)^{-1}P_k(x, \epsilon)Q_{k,0}(x, \epsilon),$$

hence, the matrix

$$(6.20) \quad \exp\left\{-\frac{x^2}{2\epsilon}J\right\}L(\epsilon)\exp\left\{\frac{x^2}{2\epsilon}J\right\} - 1_2$$

is asymptotically zero as $\epsilon \rightarrow 0$ in $S_k \cap S_j$ uniformly in the domain

(6.14-0), where 1_2 is the two-by-two identity matrix.

In the same way (manipulating with the connection formulas (6.11-0) and (6.11-1)), we can prove that the matrix

$$(6.21) \quad \exp\left\{\frac{x^2}{2\epsilon}J\right\}L_1(\epsilon)\exp\left\{-\frac{x^2}{2\epsilon}J\right\} - 1_2$$

is asymptotically zero as $\epsilon \rightarrow 0$ in $S_k \cap S_j$ uniformly in the domain

(6.14-1), where

$$(6.22) \quad L_1(\epsilon) = C(p + \delta_j(\epsilon))L(\epsilon)C(p + \delta_k(\epsilon))^{-1}.$$

Also, the matrix

$$(6.23) \quad \exp\left\{\frac{x^2}{2\epsilon}J\right\}L_2(\epsilon)\exp\left\{-\frac{x^2}{2\epsilon}J\right\} - 1_2$$

is asymptotically zero as $\epsilon \rightarrow 0$ in $S_k \cap S_j$ uniformly in the domain

(6.14-(1-1)), where

$$(6.24) \quad L_2(\epsilon) = C(-p - \delta_j(\epsilon))^{-1}L(\epsilon)C(-p - \delta_k(\epsilon)).$$

Set

$$(6.25) \quad \left\{ \begin{array}{l} L(\epsilon) = \begin{bmatrix} c_{11}(\epsilon) & c_{12}(\epsilon) \\ c_{21}(\epsilon) & c_{22}(\epsilon) \end{bmatrix}, \\ L_1(\epsilon) = \begin{bmatrix} \hat{c}_{11}(\epsilon) & \hat{c}_{12}(\epsilon) \\ \hat{c}_{21}(\epsilon) & \hat{c}_{22}(\epsilon) \end{bmatrix}, \\ L_2(\epsilon) = \begin{bmatrix} \tilde{c}_{11}(\epsilon) & \tilde{c}_{12}(\epsilon) \\ \tilde{c}_{21}(\epsilon) & \tilde{c}_{22}(\epsilon) \end{bmatrix}. \end{array} \right.$$

Then,

$$(6.26) \quad \hat{c}_{12}(\epsilon) = \{ \lambda_1(p + \delta_j(\epsilon))c_{11}(\epsilon) + c_{21}(\epsilon) \} / \lambda_2(p + \delta_\ell(\epsilon)) \\ - \lambda_1(p + \delta_\ell(\epsilon)) \{ \lambda_1(p + \delta_j(\epsilon))c_{12}(\epsilon) + c_{22}(\epsilon) \} / \lambda_2(p + \delta_\ell(\epsilon)) ;$$

and

$$(6.27) \quad \tilde{c}_{21}(\epsilon) = \lambda_1(-p - \delta_\ell(\epsilon))c_{11}(\epsilon) + \lambda_2(-p - \delta_\ell(\epsilon))c_{12}(\epsilon) \\ - \frac{\lambda_1(-p - \delta_j(\epsilon))}{\lambda_2(-p - \delta_j(\epsilon))} \{ \lambda_1(-p - \delta_\ell(\epsilon))c_{21}(\epsilon) + \lambda_2(-p - \delta_\ell(\epsilon))c_{22}(\epsilon) \}.$$

Utilizing the fact that, for any $\epsilon \in S_\ell \cap S_j$, there exists an $x \in D_2$ such that

(i) (x, ϵ) is in the domain (6.14-h),

(ii) $x/\epsilon^{\frac{1}{2}}$ takes either a real value or a purely imaginary value,

we derive from (6.20), (6.21) and (6.23) that

(1) $c_{11}(\epsilon) - 1$ and $c_{22}(\epsilon) - 1$ are asymptotically zero as $\epsilon \rightarrow 0$ in $S_\ell \cap S_j$;

(2) $|c_{12}(\epsilon)| \leq c \exp(-r^2/|\epsilon|)$, $|c_{21}(\epsilon)| \leq c \exp(-r^2/|\epsilon|)$

for $\epsilon \in S_\ell \cap S_j$, where c is a positive constant;

(3) $|\hat{c}_{12}(\epsilon)| \leq c \exp(-r^2/|\epsilon|)$ for $\epsilon \in S_\ell \cap S_j$;

(4) $|\tilde{c}_{21}(\epsilon)| \leq c \exp(-r^2/|\epsilon|)$ for $\epsilon \in S_\ell \cap S_j$.

Set $\mu(a) = \lambda_1(a)/\lambda_2(s)$. Then,

$$\begin{aligned} & |\mu(-p - \delta_i(\epsilon))c_{11}(\epsilon) - \mu(-p - \delta_j(\epsilon))c_{22}(\epsilon)| \\ &= |\mu(-p - \delta_i(\epsilon))(c_{11}(\epsilon) - c_{22}(\epsilon)) + \{\mu(-p - \delta_i(\epsilon)) - \mu(-p - \delta_j(\epsilon))\}c_{22}(\epsilon)| \\ &\leq \tilde{c} \exp(-r^2/|\epsilon|) \quad \text{in } S_i \cap S_j \end{aligned}$$

for some $\tilde{c} > 0$. Since $\mu(-p) \neq 0$, we have

$$(6.28) \quad |c_{11}(\epsilon) - c_{22}(\epsilon)| \leq c_1 |\delta_i(\epsilon) - \delta_j(\epsilon)| + c_2 \exp(-r^2/|\epsilon|)$$

in $S_i \cap S_j$ for some $c_1 > 0$ and $c_2 > 0$. On the other hand,

$$\begin{aligned} & |\lambda_1(p + \delta_j(\epsilon))c_{11}(\epsilon) - \lambda_1(p + \delta_i(\epsilon))c_{22}(\epsilon)| \\ &= |(\lambda_1(p + \delta_j(\epsilon)) - \lambda_1(p + \delta_i(\epsilon)))c_{11}(\epsilon) + \lambda_1(p + \delta_i(\epsilon))(c_{11}(\epsilon) - c_{22}(\epsilon))| \\ &\leq \hat{c} \exp(-r^2/|\epsilon|) \quad \text{in } S_i \cap S_j \end{aligned}$$

for some $\hat{c} > 0$. Since $\lambda_1(p) = 0$ and $\frac{d\lambda_1}{da}(p) \neq 0$, we have

$$\begin{aligned} (6.29) \quad |\delta_i(\epsilon) - \delta_j(\epsilon)| &\leq c_3 |\lambda_1(p + \delta_i(\epsilon))| |c_{11}(\epsilon) - c_{22}(\epsilon)| \\ &\quad + c_4 \exp(-r^2/|\epsilon|) \quad \text{in } S_i \cap S_j \end{aligned}$$

for some $c_3 > 0$ and $c_4 > 0$. An estimate (6.2) follows from (6.28) and

(6.29).

7. Resonance: In this section, we shall prove Theorem 1.2. To do this, we return to Section 5. We proved there that the transformation (5.16) takes the system (5.9) to (5.17) in the domain (5.15). The function $\delta(\varepsilon)$ satisfies the condition (5.14). We replace (5.14) by

$$(7.1) \quad |\delta(\varepsilon)| \leq H \exp(-r^2/|\varepsilon|) \quad \text{in } S$$

for some positive number H .

Set

$$(7.2) \quad \begin{cases} \phi_h(x, \varepsilon) = \Lambda(\varepsilon) \Psi_h(x/\varepsilon^{\frac{1}{2}}, p + \delta(\varepsilon)) , \\ \tilde{\phi}_h(x, \varepsilon) = \Lambda(\varepsilon) \Psi_h(x/\varepsilon^{\frac{1}{2}}, p) , \end{cases} \quad (h = -1, 0, 1, 2) .$$

Then, $\phi_h(x, \varepsilon)$ (resp. $\tilde{\phi}_h(x, \varepsilon)$) are fundamental matrix solutions of (5.17) (resp. (5.10)) such that

$$(7.3) \quad \begin{cases} \phi_0(x, \varepsilon) = \phi_1(x, \varepsilon) C(p + \delta(\varepsilon)) , \\ \phi_1(x, \varepsilon) = \phi_2(x, \varepsilon) C(-p - \delta(\varepsilon)) , \\ \phi_2(x, \varepsilon) = \phi_{-1}(x, \varepsilon) C(p + \delta(\varepsilon)) , \\ \phi_{-1}(x, \varepsilon) = \phi_0(x, \varepsilon) C(-p - \delta(\varepsilon)) , \end{cases}$$

and

$$(7.4) \quad \begin{cases} \tilde{\phi}_0(x, \varepsilon) = \tilde{\phi}_1(x, \varepsilon) C(p) , \\ \tilde{\phi}_1(x, \varepsilon) = \tilde{\phi}_2(x, \varepsilon) C(-p) , \\ \tilde{\phi}_2(x, \varepsilon) = \tilde{\phi}_{-1}(x, \varepsilon) C(p) , \\ \tilde{\phi}_{-1}(x, \varepsilon) = \tilde{\phi}_0(x, \varepsilon) C(-p) . \end{cases}$$

Set

$$(7.5) \quad S(x, \varepsilon) = \phi_0(x, \varepsilon) \tilde{\phi}_0(x, \varepsilon)^{-1} .$$

Then, the transformation

$$(7.6) \quad V = S(x, \varepsilon) U$$

takes (5.17) to (5.10). Hence, the main part of the proof is to show that

$S(x, \varepsilon) - I_2$ is asymptotically zero as $\varepsilon \rightarrow 0$ in S uniformly in D_1 . Note

that $r_1 = r$. To do this we manipulate in a way similar to the argument in Section 6, utilizing the fact that

- (i) $\phi(\varepsilon)\exp(x^2/\varepsilon)$ and $\psi(\varepsilon)\exp(-x^2/\varepsilon)$ are asymptotically zero as $\varepsilon \rightarrow 0$ in S uniformly in D_1 ;
- (ii) $C(p) + \phi(\varepsilon)C(p)^{-1} - I_2 = O(\delta(\varepsilon))$;
- (iii) $C(-p) + \psi(\varepsilon)C(-p)^{-1} - I_2 = O(\delta(\varepsilon))$.

The details are left to the reader.

8. Proof of Theorem 1.1: We shall prove Theorem 1.1 in the case when $v = 3$. The general case can be treated in the same manner. We shall consider three sectors S_1, S_2, S_3 as shown in Figure 1.

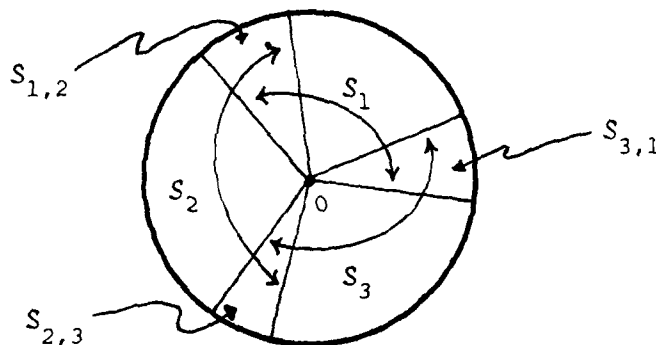


Figure 1

We denote by $S_{1,2}, S_{2,3}, S_{3,1}$ the intersections $S_1 \cap S_2, S_2 \cap S_3, S_3 \cap S_1$, respectively.

The three functions $\delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon)$ are holomorphic in S_1, S_2, S_3 , respectively. Furthermore,

$$(8.1) \quad \delta_j(\varepsilon) \text{ is asymptotically zero as } \varepsilon \rightarrow 0 \text{ in } S_j,$$

and

$$(8.2) \quad |\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)| \leq c_0 \exp(-c_1/|\varepsilon|^\lambda) \text{ in } S_{j,j+1},$$

where c_0, c_1, λ are positive numbers, and $S_{3,4} = S_{3,1}, \delta_4 = \delta_1$. We shall denote $\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)$ by $\sigma_j(\varepsilon)$.

We consider a sufficiently small disk:

$$(8.3) \quad D = \{\varepsilon; |\varepsilon| \leq \rho_0\}.$$

We choose three line-segments ℓ_1, ℓ_2, ℓ_3 starting from $\varepsilon = 0$ in such a way that

$$(8.4) \quad \ell_j \subset S_{j,j+1}. \quad (\text{Cf. Figure 2.})$$

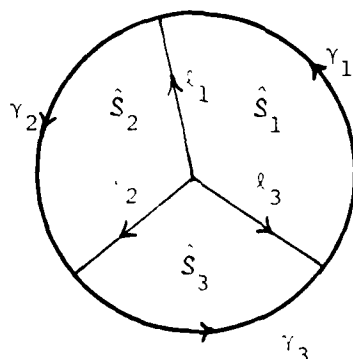


Figure 2

Three line-segments l_1, l_2, l_3 divide the disk D (cf. (8.3)) into three open sectors $\hat{S}_1, \hat{S}_2, \hat{S}_3$ (cf. Figure 2). The boundaries of $\hat{S}_1, \hat{S}_2, \hat{S}_3$ are respectively

$$(8.5-j) \quad l_{j-1} + \gamma_j - l_j, \quad j = 1, 2, 3,$$

where $l_0 = l_3$, and the γ 's are circular arcs such that

$$(8.6) \quad \gamma_1 + \gamma_2 + \gamma_3 = C = \{\varepsilon; |\varepsilon| = \rho_0\}.$$

The line-segments l_j and the circular arcs γ_j are oriented as indicated in Figure 2. We assume that ρ_0 is so small that

$$(8.7) \quad \bar{\hat{S}}_j \subset S_j,$$

where $\bar{\hat{S}}_j$ denotes the closure of \hat{S}_j .

Set, for $\varepsilon \in \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3$,

$$(8.8) \quad \delta(\varepsilon) = \delta_j(\varepsilon) \quad \text{if } \varepsilon \in \hat{S}_j.$$

Since

$$\frac{1}{2\pi i} \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(\varepsilon)}{\varepsilon - \varepsilon} d\varepsilon = \begin{cases} \delta_j(\varepsilon) & \varepsilon \in \hat{S}_j, \\ 0 & \varepsilon \notin \hat{S}_j. \end{cases}$$

we have

$$\delta(\epsilon) = \frac{1}{2\pi i} \sum_{j=1}^3 \int_{\ell_{j-1} + \gamma_j - \ell_j} \frac{\delta_j(\epsilon)}{\xi - \epsilon} d\xi \quad \text{in } \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3.$$

Utilizing

$$\frac{1}{\xi - \epsilon} = \sum_{m=0}^N \xi^{-(m+1)} \epsilon^m + \frac{\epsilon^{N+1}}{\xi^{N+1}(\xi - \epsilon)},$$

we derive

$$\begin{aligned} \delta(\epsilon) = & \frac{1}{2\pi i} \sum_{m=0}^N \left\{ \sum_{j=1}^3 \int_{\ell_{j-1} + \gamma_j - \ell_j} \xi^{-(m+1)} \delta_j(\xi) d\xi \right\} \epsilon^m \\ & + \left\{ \frac{1}{2\pi i} \sum_{j=1}^3 \int_{\ell_{j-1} + \gamma_j - \ell_j} \frac{\delta_j(\xi)}{\xi^{N+1}(\xi - \epsilon)} d\xi \right\} \epsilon^{N+1}. \end{aligned}$$

Since $\delta(\epsilon)$ is asymptotically zero as $\epsilon \rightarrow 0$ in $\hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3$, the first term must be zero, and hence

$$\delta(\epsilon) = \left\{ \frac{1}{2\pi i} \sum_{j=1}^3 \int_{\ell_{j-1} + \gamma_j - \ell_j} \frac{\delta_j(\epsilon)}{\xi^{N+1}(\xi - \epsilon)} d\xi \right\} \epsilon^{N+1}.$$

Thus we arrive at the following formula:

$$(8.9) \quad \delta(\epsilon) = \frac{1}{2\pi i} \left\{ \sum_{j=1}^3 \int_{\ell_j} \frac{\sigma_j(\xi)}{\xi^N(\xi - \epsilon)} d\xi + \int_C \frac{\delta(\xi)}{\xi^N(\xi - \epsilon)} d\xi \right\} \epsilon^N$$

for $\epsilon \in \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3$ and $N = 1, 2, 3, \dots$, where $\sigma_j = \delta_{j+1} - \delta_j$.

Construct three open sectors $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3$ as shown in Figure 3, where $0 < \rho_1 < \rho_0$ and θ is a small positive number. Then,

$$\left| \int_C \frac{\delta(\xi)}{\xi^N(\xi - \epsilon)} d\xi \right| \leq \frac{C_0}{\rho_0^{N-1}} \frac{1}{\rho_0 - \rho_1}$$

and

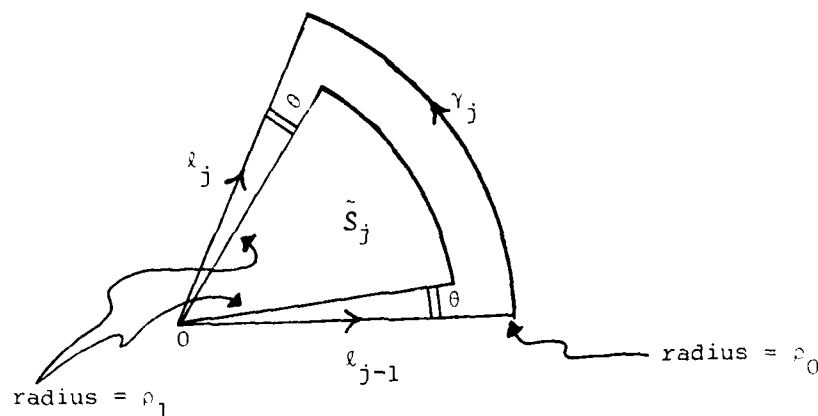


Figure 3

$$\begin{aligned}
 \left| \int_{\tilde{S}_j} \frac{\sigma_j(\xi)}{\xi^N (\xi - \epsilon)} d\xi \right| &\leq \frac{c_0}{\sin \theta} \int_0^{\rho_0} t^{-N-1} \exp(-c_1 t^{-\lambda}) dt \\
 &< \frac{c_0}{\lambda \sin \theta} \int_0^{+\infty} \tau^{(N\lambda^{-1}-1)} \exp(-c_1 \tau) d\tau \\
 &= \frac{c_0}{\lambda \sin \theta} c_1^{-(N/\lambda)} \Gamma(N/\lambda)
 \end{aligned}$$

for $\epsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3$, where c_0 is a positive number. Since

$$\Gamma(N/\lambda) \leq c_1 (N/\lambda)^{(N/\lambda)} e^{-(N/\lambda)}$$

for some $c_1 > 0$, we have

$$(8.10) \quad |\delta(\epsilon)| \leq c_2 \left(\frac{|\epsilon|^\lambda N}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)}$$

for $\epsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \tilde{S}_3$; c_1 is a positive number. For a given ϵ , choose N so that

$$\frac{N}{\lambda} < \frac{c_1}{|\epsilon|^\lambda} \leq \frac{N+1}{\lambda}.$$

Then, it follows from (8.10) that

$$(8.11) \quad |\delta(\epsilon)| \leq c_2 e^{\frac{1}{\lambda}} \exp(-c_1/|\epsilon|^\lambda).$$

choosing c_1, c_2, c_3 in various ways, we can complete the proof of Theorem 1.1.

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20. ABSTRACT - Cont'd.

c_0, c_1 and λ whenever $S_j \cap S_k \neq \emptyset$, we prove that $|\delta_j(\epsilon)| \leq c_2 \exp\{-c_1/|\epsilon|^2\}$ in S_j for some positive number c_2 . Then, utilizing this result, we prove that Matkowsky-condition implies the resonance in the sense of N. Kopell under a reasonable assumption. The sufficiency of Matkowsky-condition with regard to the Ackerberg-O'Malley resonance has been an open question. This work gives an affirmative answer to this question in a reasonably general case.